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Free-Wave Propagation Relationships of Second-Order and Fourth-Order Periodic Systems

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PREFACE

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14. ABSTRACT

This report develops an analytical expression for the determinant of two diagonally-indexed, full matrices when they are zero. These matrices originate from second- and fourth-order periodic system theory. The partial differential equations of these systems are solved using a series solution and are converted into closed-form analytical expressions. The denominators of these expressions are zero when free-wave propagation is present, and these denominators are equated to the determinants of the system matrices derived from a second analytical method. This process develops a relationship between frequency and wavenumber that is explicit for free-wave propagation in these systems. Two examples are included to illustrate this new method.

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1. INTRODUCTION

Periodic systems are studied and modeled for use in different structures for various rationales. Some examples are reinforced ship hulls, railroad tracks, aircraft fuselages, and trolley lines. Analysis of these systems has been ongoing for many years. Some of the analysis can be extremely complex (reference 1) and can produce a solution to a specific problem, and some of the analysis can be simple (reference 2) and produces insight into system behavior. These problems are sometimes formulated to determine the free-wave propagation of a structure and sometimes to calculate the forced wave response to a specific type of load. For simple mechanical systems, these problems can be broken down into two specific classes:

1. Second-order systems (such as bars and strings) where the governing differential equation contains a second-order derivative with respect to the spatial variable.

2. Fourth-order systems (such as plates and beams) where the governing differential equation contains a fourth-order derivative with respect to the spatial variable.

Models of second order systems begin with the wave equation in space and time. These systems can become periodic with the addition of stiffeners (springs) and/or masses (beads). These systems have been studied with multiple sets of stiffeners and masses (reference 3) to determine the dynamic response. The tensioned string with periodic stiffeners has been analyzed in the frequency domain with moving harmonic forces (reference 4) and a suddenly-applied concentrated force (reference 5). These papers discuss the stability of the system, particularly with respect to the value of the stiffeners, although there is no extension of the problem into the wavenumber domain. The response in a nearly-periodic beaded string has been analyzed using a matrix method (reference 6). A model and experiment of a nearly-periodic pendulum system has also been formulated (reference 7) with an emphasis on Anderson localization.

Models of fourth-order systems typically begin with the Euler-Bernoulli beam (or plate) equation in space and time. These systems can become periodic with the addition of stiffeners (springs) and/or masses (beads). In reference 8, a fourth-order system was studied for free-wave propagation and a distributed pressure load; a spatial domain solution was constructed and a contour integral was evaluated to transfer the expression into the wavenumber domain. This work contains an expression for free-wave propagation in the reinforced plate. The work in reference 8 was extended to stiffened plates with line and point forces (reference 9). Work has also been completed on the modeling of plates that have sectional aperiodicity (reference 10).

The present report develops a relationship between the determinant of a periodic system dynamic matrix and an analytical expression. Specifically, the case of free-wave motion is investigated in the wavenumber frequency domain. Two separate methods are used to solve the periodic problem. First, a series solution to the problem is generated and this is converted into a closed form analytical expression using infinite series of residues. Next, a matrix solution is formulated. By equating functions of the two methods for free-wave propagation, an analytical expression is derived when the determinant of the system matrix is zero. This expression is useful for understanding the behavior of these diagonally-indexed, full matrices. This method is applied to both a second-order and fourth order system. The origin of each individual problem is discussed, and two numerical examples are illustrated.

2. SECOND-ORDER SYSTEM

The motion of a reinforced string is a second-order system that is governed by the wave equation. If the stiffeners reinforcing the string are equally spaced, and if the string has infinite spatial extent, then the partial differential equation modeling this motion in the spatial-time domain is

$$T\frac{\partial^2 \hat{w}(x,t)}{\partial x^2} - \rho\frac{\partial^2 \hat{w}(x,t)}{\partial t^2} = \hat{f}(x,t) + K\sum_{n=-\infty}^{n=+\infty} \hat{w}(x,t)\,\delta(x-nL)\ , \tag{1}$$

where

- $\hat{w}(x,t)$ is the transverse displacement in the y-direction,
- x is the spatial location on the string,
- t is time,
- T is the tension,
- ρ is the mass per unit length,
- $\hat{f}(x,t)$ is the external load on the string per unit length,
- K is the stiffness of each reinforcing member,
- L is the distance between adjacent stiffeners, and
- δ is the Dirac delta function.

Using a point load that is harmonic in time, and transferring equation (1) into the wavenumber-frequency domain yields (reference 3)

$$(k^2 - k_t^2)w(k) = \frac{-F_0}{T} - \frac{K}{TL}\sum_{n=-\infty}^{n=+\infty} w\left(k + \frac{2\pi n}{L}\right), \tag{2}$$

where $w(k)$ is the transverse displacement in the y-direction, k is the wavenumber, F_0 is the magnitude of an applied point force at $x = 0$, and k_t is the free wavenumber of the unreinforced string and is given by

$$k_t = \frac{\omega}{c_t} = \frac{\omega}{\sqrt{T/\rho}} \ , \tag{3}$$

where ω is the frequency and c_t is the free-wave speed of the unreinforced string. Note that the free wavenumber implicitly contains the frequency. The solution to this equation has been previously solved and is given by

$$\frac{w(k)}{F_0} = \frac{-1}{(k^2 - k_t^2)\left[T + \frac{K}{L}\sum_{n=-\infty}^{n=+\infty}(k_n^2 - k_t^2)^{-1}\right]} \ , \tag{4}$$

where $k_n = k + 2\pi n / L$ and it is noted that $k_0 \equiv k$. The solution given in equation (4) is now converted from a series solution into a closed-form solution using an analytical expression for the summation in the denominator. Using the analytical expression given by (reference 11)

$$\sum_{n=-\infty}^{n=+\infty}(k_n^2 - k_t^2)^{-1} = \frac{L\cos\left(\frac{k_t L}{2}\right)\sin\left(\frac{k_t L}{2}\right)}{2k_t\left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{k_t L}{2}\right)\right]} \ . \tag{5}$$

Equation (5) can be inserted into equation (4), and this yields the system response:

$$\frac{w(k)}{F_0} = \frac{\sin\left[\frac{(k+k_t)L}{2}\right]\sin\left[\frac{(k-k_t)L}{2}\right]}{\left(k^2 - k_t^2\right)\left\{T\left[\sin^2\left(\frac{k_t L}{2}\right) - \sin^2\left(\frac{kL}{2}\right)\right] - \left(\frac{K}{2k_t}\right)\cos\left(\frac{k_t L}{2}\right)\sin\left(\frac{k_t L}{2}\right)\right\}} \ . \tag{6}$$

For free-wave propagation to exist in this system, the denominator of equation (6) must be zero, i.e.,

$$T\left[\sin^2\left(\frac{k_t L}{2}\right) - \sin^2\left(\frac{kL}{2}\right)\right] - \left(\frac{K}{2k_t}\right)\cos\left(\frac{k_t L}{2}\right)\sin\left(\frac{k_t L}{2}\right) = 0 \ . \tag{7}$$

The term $k^2 - k_t^2$ is not a free-wave propagation condition for the reinforced system because when $k^2 = k_t^2$ this term will cancel with one of the terms in the numerator of equation (6). The relationship between wavenumber and free wavenumber (and thus frequency) for free-wave propagation for the reinforced system is

$$\sin\left(\frac{kL}{2}\right) = \sqrt{\sin^2\left(\frac{k_t L}{2}\right) - \frac{K}{2Tk_t}\cos\left(\frac{k_t L}{2}\right)\sin\left(\frac{k_t L}{2}\right)} = \theta \ , \tag{8}$$

and this yields the multi-valued solution to the wavenumber, written as

$$k = \pm\left|\frac{2}{L}\arcsin(\theta) + \frac{2n\pi}{L}\right| \qquad n = \cdots -2,-1,0,1,2\cdots \ , \tag{9}$$

provided that

$$-1 \leq \theta \leq 1 \ ,$$

which can be alternatively stated as

$$\tan\left(\frac{k_t L}{2}\right) \geq \frac{K}{2Tk_t} \ . \tag{10}$$

Another method to solve this problem can be constructed based on the spatial periodicity. Equation (2) can be changed using $k_n = k + 2\pi\, n / L$ so that an infinite number of indexed equations are written as

$$\begin{gathered}
\vdots \\
(k_{-1}^2 - k_t^2) w_{-1}(k_{-1}) = \frac{-F_0}{T} - \frac{K}{TL}\sum_{n=-\infty}^{n=+\infty} w_n(k_n) \\
(k_0^2 - k_t^2) w_0(k_0) = \frac{-F_0}{T} - \frac{K}{TL}\sum_{n=-\infty}^{n=+\infty} w_n(k_n) \\
(k_1^2 - k_t^2) w_1(k_1) = \frac{-F_0}{T} - \frac{K}{TL}\sum_{n=-\infty}^{n=+\infty} w_n(k_n) \\
\vdots
\end{gathered} \ . \tag{11}$$

These individual equations can be placed into matrix format as

$$\left[\ [\mathbf{D}] + \left(\frac{K}{TL}\right)[\mathbf{1}]\ \right]\{\mathbf{w}\} = \{\mathbf{f}\} \ , \tag{12}$$

where $[\mathbf{D}]$ is a diagonal matrix whose entries are equal to $(k_n^2 - k_t^2)$, $[\mathbf{1}]$ is a matrix whose entries are all 1, $\{\mathbf{w}\}$ is a vector of the unknown displacements, and $\{\mathbf{f}\}$ is a vector of all ones multiplied by $-F_0/T$. The matrix in equation (12) can be truncated to a finite number of terms and solved, yielding

$$\{\mathbf{w}\} = \left[\ [\mathbf{D}] + \left(\frac{K}{TL}\right)[\mathbf{1}]\ \right]^{-1}\{\mathbf{f}\} \ , \tag{13}$$

Note that onee equation (13) is solved, the displaeement term that corresponds to the system's displacement in the wavenumber domain (equation (6)) is $w_0(k_0)$. If enough terms are chosen in equation (13), this solution will converge on the expression in equation (6). For the system presented in equation (12) to support free wave propagation, it is necessary that

$$\det\left\{[\mathbf{D}]+\left(\frac{K}{TL}\right)[\mathbf{1}]\right\}=0\ . \tag{14}$$

Therefore, the relationship between the determinant in equation (14) and the system's free-wave propagation requires that

$$\det\begin{bmatrix} \ddots & & \vdots & & \iddots \\ & \left(k_{-1}^2-k_t^2+\dfrac{K}{TL}\right) & \dfrac{K}{TL} & \dfrac{K}{TL} & \\ \cdots & \dfrac{K}{TL} & \left(k_0^2-k_t^2+\dfrac{K}{TL}\right) & \dfrac{K}{TL} & \cdots \\ & \dfrac{K}{TL} & \dfrac{K}{TL} & \left(k_1^2-k_t^2+\dfrac{K}{TL}\right) & \\ \iddots & & \vdots & & \ddots \end{bmatrix}=0\ , \tag{15}$$

if, and only if,

$$\sin\left(\frac{kL}{2}\right)=\sqrt{\sin^2\left(\frac{k_tL}{2}\right)-\frac{K}{2Tk_t}\mathrm{eos}\left(\frac{k_tL}{2}\right)\sin\left(\frac{k_tL}{2}\right)}\ , \tag{16}$$

This relationship is illustrated graphieally in figure 1. For this example, the frequency is 275 Hz, the tension is 100 N, the mass per unit length is 1 kg m^{-1}, the stiffener spring constant is 30,000 N/m, the stiffener spaeing is 0.05 m, and 37 terms were used to eonstruet the determinant. The solid line in this figure is a plot of the determinant of $\{[\mathbf{D}]+[K/(TL)][\mathbf{1}]\}$ in the decibel seale versus wavenumber from -600 rad/m to 600 rad/m. The solid black dots are the free-wave propagation wavenumbers calculated from equation (9), and these mateh the minimum values of the determinant with respect to the wavenumber.

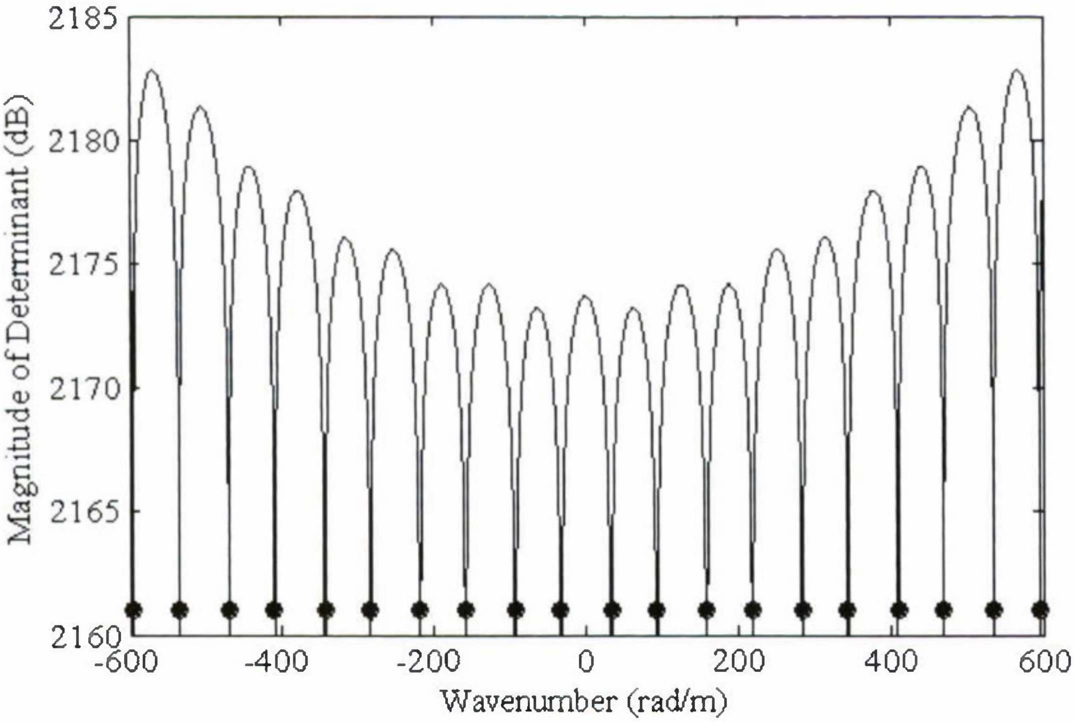

Figure 1. Magnitude of Determinant vs Wavenumber for a Second-Order System (solid line) and Free-Wave Propagation Wavenumbers (dots) Calculated from Equation (9)

3. FOURTH-ORDER SYSTEM

The motion of a thin, reinforced plate is a fourth-order system that is governed by the Euler-Bernoulli plate equation. If the reinforcing ribs are equally spaced, and if the plate has infinite spatial extent, then the partial differential equation modeling of this motion in the spatial-time domain is

$$D\frac{\partial^4 \hat{w}(x,t)}{\partial x^4} + \rho h\frac{\partial^2 \hat{w}(x,t)}{\partial t^2} = -\hat{f}(x,t) - K\sum_{n=-\infty}^{n=+\infty} \hat{w}(x,t)\,\delta(x-nL) \quad , \tag{17}$$

where $\hat{w}(x,t)$ is the transverse displacement in the y-direction, x is the spatial location on the plate, t is time, ρ is the density, h is the height of the plate, $\hat{f}(x,t)$ is the external load on the plate, K is the stiffness of each rib per unit length, L is the distance between adjacent ribs, and δ is the Dirac delta function. The flexural rigidity of the plate D is

$$D = \frac{Eh^3}{12(1-\upsilon^2)} \quad , \tag{18}$$

where υ is Poisson's ratio and E is Young's modulus of the plate. Using a line load that is harmonic in time, and transferring equation (17) into the wavenumber-frequency domain, yields (reference 10)

$$(k^4 - k_f^4)w(k) = \frac{-F_0}{D} - \frac{K}{DL}\sum_{n=-\infty}^{n=+\infty} w\left(k + \frac{2\pi n}{L}\right), \tag{19}$$

where $w(k)$ is the transverse displacement in the y-direction, k is the wavenumber, F_0 is the magnitude of the applied line force at $x = 0$, and k_f is the free (flexural) wavenumber of the unreinforced plate and is given by

$$k_f = \left(\frac{\rho h \omega^2}{D}\right)^{1/4}, \tag{20}$$

where ω is the frequency. The solution to this equation has been previously solved and is given by

$$\frac{w(k)}{F_0} = \frac{-1}{(k^4 - k_f^4)\left[D + \frac{K}{L}\sum_{n=-\infty}^{n=+\infty}(k_n^4 - k_f^4)^{-1}\right]}. \tag{21}$$

The solution given in equation (21) is now converted from a series solution into a closed form solution using an analytical expression for the summation in the denominator. Using the analytical expression given by reference 11

$$\sum_{n=-\infty}^{n=+\infty}(k_n^4 - k_f^4)^{-1} =$$

$$\frac{L\left\{\sin(k_f L)\left[\cos^2\left(\frac{kL}{2}\right) - \cosh^2\left(\frac{k_f L}{2}\right)\right] + \sinh(k_f L)\left[\cos^2\left(\frac{k_f L}{2}\right) - \cos^2\left(\frac{kL}{2}\right)\right]\right\}}{8k_f^3\left[\cos^2\left(\frac{k_f L}{2}\right) - \cos^2\left(\frac{kL}{2}\right)\right]\left[\cos^2\left(\frac{kL}{2}\right) - \cosh^2\left(\frac{k_f L}{2}\right)\right]}. \tag{22}$$

Equation (22) can be inserted into equation (21), and this yields the system response:

$$\frac{w(k)}{F_0} = \frac{-1}{(k^4 - k_f^4)}\left(\frac{\Phi}{\Delta}\right), \tag{23}$$

where

$$\Phi = 8k_f^3\left[\cos^2\left(\frac{k_f L}{2}\right) - \cos^2\left(\frac{kL}{2}\right)\right]\left[\cos^2\left(\frac{kL}{2}\right) - \cosh^2\left(\frac{k_f L}{2}\right)\right], \tag{24}$$

and

$$\Delta = K\left\{\sin(k_f L)\left[\cos^2\left(\frac{kL}{2}\right) - \cosh^2\left(\frac{k_f L}{2}\right)\right] + \sinh(k_f L)\left[\cos^2\left(\frac{k_f L}{2}\right) - \cos^2\left(\frac{kL}{2}\right)\right]\right\}$$
$$+ 8Dk_f^3\left[\cos^2\left(\frac{k_f L}{2}\right) - \cos^2\left(\frac{kL}{2}\right)\right]\left[\cos^2\left(\frac{kL}{2}\right) - \cosh^2\left(\frac{k_f L}{2}\right)\right] \quad . \qquad (25)$$

For free-wave propagation to exist in this system, the denominator of equation (23) must be zero, i.e.,

$$\Delta = 0 \ , \qquad (26)$$

where Δ is rewritten as

$$\Delta = a\cos^4\left(\frac{kL}{2}\right) + b\cos^2\left(\frac{kL}{2}\right) + c \ , \qquad (27)$$

with

$$a = -8Dk_f^3 \ , \qquad (28)$$

$$b = K\sin\left(k_f L\right) - K\sinh\left(k_f L\right) + 8Dk_f^3\cos^2\left(\frac{k_f L}{2}\right) + 8Dk_f^3\cosh^2\left(\frac{k_f L}{2}\right) \ , \qquad (29)$$

and

$$c = K\sinh\left(k_f L\right)\cos^2\left(\frac{k_f L}{2}\right) - K\sin\left(k_f L\right)\cosh^2\left(\frac{k_f L}{2}\right)$$
$$- 8Dk_f^3\cos^2\left(\frac{k_f L}{2}\right)\cosh^2\left(\frac{k_f L}{2}\right). \qquad (30)$$

It is noted that Mace (reference 8) developed a similar expression for this relationship, although his parameter set was non-dimensional. The term $k^4 - k_f^4$ is not a free-wave propagation condition for the reinforced system because when $k^4 = k_f^4$ this term will cancel with one of the terms in the numerator in equation (23). Equation (27) yields a system with free-wave propagation whenever

$$\cos\left(\frac{kL}{2}\right) = \pm\sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} = \phi \ , \qquad (31)$$

provided that

$$-1 \le \phi \le 1 \ . \tag{32}$$

Numerical simulations suggest that the second root that contains a minus sign inside the radical is extraneous. If the conditions listed in equation (32) are satisfied, then the wavenumber where the free-wave propagation occurs can be calculated using the formula

$$k = \pm \left| \frac{2}{L} \arccos(\phi) + \frac{2n\pi}{L} \right| \qquad n = \cdots -2, -1, 0, 1, 2 \cdots \ . \tag{33}$$

If the condition listed in equation (32) is not satisfied, then the system does not support free-wave propagation at that specific frequency.

An alternative method to solve this problem can be constructed based on the spatial periodicity using equation (19) and the relationship $k_n = k + 2\pi n / L$. This yields

$$\begin{aligned} &\vdots \\ (k_{-1}^4 - k_f^4) w_{-1}(k_{-1}) &= \frac{-F_0}{D} - \frac{K}{DL} \sum_{n=-\infty}^{n=+\infty} w_n(k_n) \\ (Dk^4 - k_f^4) w_0(k_0) &= \frac{-F_0}{D} - \frac{K}{DL} \sum_{n=-\infty}^{n=+\infty} w_n(k_n) \ . \\ (Dk_1^4 - k_f^4) w_1(k_1) &= \frac{-F_0}{D} - \frac{K}{DL} \sum_{n=-\infty}^{n=+\infty} w_n(k_n) \\ &\vdots \end{aligned} \tag{34}$$

The individual equations from equation (34) can be placed into matrix form as

$$\left[\ [\mathbf{D}] + \left(\frac{K}{DL} \right) [\mathbf{1}] \ \right] \{\mathbf{w}\} = \{\mathbf{f}\} \ , \tag{35}$$

where $[\mathbf{D}]$ is a diagonal matrix whose entries are $(k_n^4 - k_f^4)$, $[\mathbf{1}]$ is a matrix whose entries are all 1, $\{\mathbf{w}\}$ is a vector of the unknown displacements, and $\{\mathbf{f}\}$ is a vector of all ones multiplied by $-F_0 / D$. The matrix equation in equation (35) can be truncated to a finite number of terms and solved, yielding

$$\{\mathbf{w}\} = \left[\ [\mathbf{D}] + \left(\frac{K}{DL} \right) [\mathbf{1}] \ \right]^{-1} \{\mathbf{f}\} \ . \tag{36}$$

Note that once equation (36) is solved, the displacement term that corresponds to the displacement of the system in the wavenumber domain is $w_0(k_0)$. If enough terms are chosen, this solution will approach the expression in equation (23). For the system presented in equation (36) to support free-wave propagation, it is necessary that

$$\det\left\{[\mathbf{D}]+\left(\frac{K}{DL}\right)[\mathbf{1}]\right\}=0 \ . \tag{37}$$

Therefore, the relationship between the determinant in equation (37) and free-wave propagation of the system requires that

$$\det\begin{bmatrix} \ddots & & \vdots & & \iddots \\ & \left(k_{-1}^4-k_f^4+\frac{K}{DL}\right) & \frac{K}{DL} & \frac{K}{DL} & \\ \cdots & \frac{K}{DL} & \left(k_0^4-k_f^4+\frac{K}{DL}\right) & \frac{K}{DL} & \cdots \\ & \frac{K}{DL} & \frac{K}{DL} & \left(k_1^4-k_f^4+\frac{K}{DL}\right) & \\ \iddots & & \vdots & & \ddots \end{bmatrix}=0 \tag{38}$$

if, and only if,

$$K\left\{\sin(k_fL)\left[\cos^2\left(\frac{kL}{2}\right)-\cosh^2\left(\frac{k_fL}{2}\right)\right]+\sinh(k_fL)\left[\cos^2\left(\frac{k_fL}{2}\right)-\cos^2\left(\frac{kL}{2}\right)\right]\right\}$$
$$+8Dk_f^3\left[\cos^2\left(\frac{k_fL}{2}\right)-\cos^2\left(\frac{kL}{2}\right)\right]\left[\cos^2\left(\frac{kL}{2}\right)-\cosh^2\left(\frac{k_fL}{2}\right)\right]=0 \ . \tag{39}$$

This relationship is illustrated graphically in figure 2. For this example, the following applies:

- Frequency is 100 Hz.
- Thickness of the plate is 0.005 m.
- Young's modulus of the plate is 200e9 N m^{-2}.
- Poisson's ratio of the plate is 0.3.
- Density of the plate is 7860 kg m^{-3}.
- Stiffener spring constant per unit length is 1e6 N m^{-2}.
- Stiffener spacing is 1.0 m.
- Eleven terms were used to construct the determinant.

The solid line in this figure is a plot of the determinant of $[\mathbf{D}]+[K/(DL)][\mathbf{1}]$ in the decibel scale versus wavenumber from −20 rad/m to 20 rad/m. The solid black dots are the free-wave propagation wavenumbers calculated from equation (33), and these match the minimum values of the determinant with respect to the wavenumber.

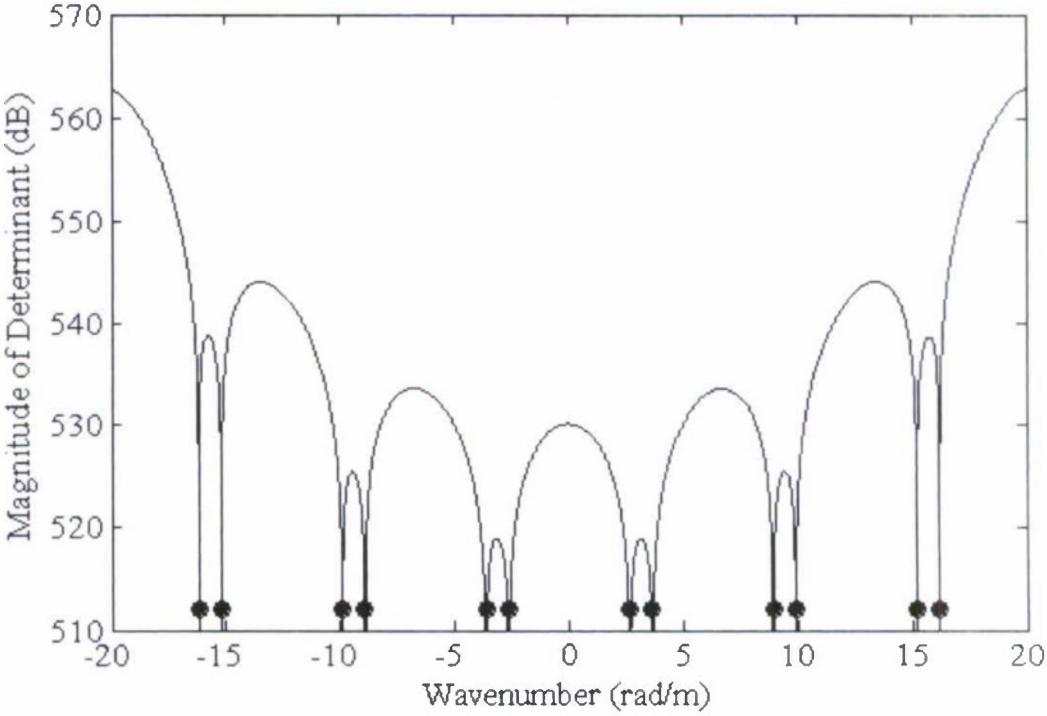

Figure 2. Magnitude of Determinant vs Wavenumber for a Fourth-Order System (solid line) and Free-Wave Propagation Wavenumbers (dots) Calculated from Equation (33)

4. SUMMARY

An analytical expression for free-wave propagation for periodic systems has been developed by solving for the dynamic behavior using two separate methods and equating the results. This process yields a relationship between a diagonally-indexed, full determinant of a large matrix when it is zero and a simplified analytical expression. This process was undertaken for both a second and fourth order system. Two example problems are included to illustrate the accuracy of this process.

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Center for Naval Analyses	1